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COMMENT

On entropy method of turbulence closure problem

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Abstract. It is proved that the turbulence entropy, defined in the (maximal) entropy method of turbulence closure problem, does not attain a maximum value for stationary turbulence, and a solution for the basic equations of the entropy method does not exist.

1. Introduction

Edwards and McComb (1969) proposed a maximal entropy principle for stationary turbulence and developed the entropy method to deal with the closure problem in turbulence theory. Leslie (1983) gave a critical account of the entropy method in his celebrated book. Recently McComb (1990) made an effort to persuade readers that the entropy method is reasonable and successful. In this comment, it is proved that the turbulence entropy does not attain a maximum value for stationary turbulence and the solution for the basic equations of the entropy method does not exist. Then some relevant issues are discussed.

2. The entropy method

The entropy of a system is

$$S = -\kappa \int P(\bar{X}) \ln P(\bar{X}) \, \mathrm{d}\bar{X}. \tag{1}$$

Here κ is the Boltzmann constant, $\bar{X} = \{X_1, X_2, X_3, \ldots\}$ denotes the state vector of the system, $X_i (i = 1, 2, 3, \ldots)$ are the state variables, and $P(\bar{X})$ is the probability distribution function. An isolated system will evolve to reach a thermodynamic equilibrium where the entropy S attains a maximum value. Edwards and McComb (1969) assume that the entropy of stationary turbulence also attains a maximum value while the variables are subject to the energy equation. Their working expression for the turbulence entropy is

$$S = S_0 + S_1 \tag{2a}$$

$$S_0 = \tfrac{1}{2}\kappa \sum_i [1 + \ln(2\pi\phi_i)] \tag{2b}$$

$$S_1 = 2\kappa \sum_{ijm} [M_{ijm} M_{jim} \phi_m (\phi_j - \phi_i)/\phi_i (\eta_i + \eta_j + \eta_m)^2]. \tag{2c}$$

Equation (2) is equation (7.99) from Leslie (1983), and corresponds to equation (7.92) in McComb (1990). Here we adopt the notation used by Leslie (1983), ϕ_i is the average modal intensity or energy, and η_i is the dynamic damping coefficient. During the derivation of

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equation (2) (chapter 7 in Leslie's book), it is implicitly assumed that the state variables X_i are real, so the nonlinear interaction coefficients M_{ijm} are also real. Of course Leslie did not give an explicit expression for real X_i and M_{ijm}. An explicit expression for the real state variables X_i of turbulence was derived by Qian (1983), and the corresponding real coefficients M_{ijm} are the A_{ijm} in Qian (1983).

By the entropy method, the response equation is given by $\delta S = 0$, or

$$\partial S/\partial \eta_i + \sum_n (\partial S/\partial \phi_n)(\delta \phi_n/\delta \eta_i) = 0 \qquad \text{for all } i \tag{3}$$

which is equation (7.100) in Leslie (1983) or equation (7.88) of McComb (1990). The term $(\delta \phi_n/\delta \eta_i)$ in (3) is to be calculated by varying the following energy equation for a stationary turbulence:

$$(\nu_i - \nu_i')\phi_i = -4 \sum_{jm} [M_{ijm} M_{jim} \phi_m (\phi_j - \phi_i)/(\eta_i + \eta_j + \eta_m)]. \tag{4a}$$

Equation (4a) is equation (7.60) from Leslie (1983). The sole application of the entropy method which has been made up to now is to derive the inertial-range spectrum. In the inertial range, $\nu_i' = \nu_i = 0$, and the energy equation (4a) is replaced by

$$\Pi(k) = \epsilon \tag{4b}$$

which is equation (6.71) in Leslie (1983) or equation (6.43) in McComb (1990). Here $\Pi(k)$ is the energy transfer function and is invariant under the transformation $\phi_i \to \lambda \phi_i$ and $\eta_i \to \lambda^2 \eta_i$, ϵ is the energy dissipation rate and is a constant. Equations (2)–(4) constitute the basic equations of the entropy method.

3. Theorem of the non-existence of the maximum of turbulence entropy

Whether the entropy method is well ground has been a controversial issue. Leslie (1983) expressed much doubt about it, and pointed out: 'it seems most improbable that the entropy will actually attain a maximum value if there are forces which prevent the system from reaching equilibrium.' McComb (1990) made an effort to defend the entropy method and argued that the entropy method is reasonable and successful. In this section, we will prove that the turbulence entropy does not have a maximum value under constraint (4a) or (4b), and the solution for the basic equations (2)–(4) of the entropy method does not exist. First, we prove the following lemma.

Lemma. S_1 given in (2c) is negative, i.e. $S_1 < 0$.

Proof. Let $W_{ijm} = (M_{ijm}\phi_j\phi_m + M_{jmi}\phi_m\phi_i + M_{mij}\phi_i\phi_j)/(\eta_i + \eta_j + \eta_m)$. By using the properties of M_{ijm}, from (2c) we have

$$\begin{aligned} S_1 &= -\kappa \sum_{ijm} [M_{ijm} W_{ijm}/\phi_i(\eta_i + \eta_j + \eta_m)] \\ &= -\kappa \sum_{ijm} [M_{ijm}\phi_j\phi_m W_{ijm}/\phi_i\phi_j\phi_m(\eta_i + \eta_j + \eta_m)] \\ &= -\tfrac{1}{3}\kappa \sum_{ijm} [W_{ijm}^2/\phi_i\phi_j\phi_m] < 0. \end{aligned} \tag{5}$$

As mentioned before, the M_{ijm} are real, so the W_{ijm} are real. Actually we can prove the lemma in another way. To a first-order approximation, $P(\boldsymbol{X}) = P^{(F)} + P^{(1)}$, $P^{(F)}$ is the Gaussian function and $P^{(1)}$ is the first-order correction. $P(\boldsymbol{X})$ and $P^{(F)}$ are positive numbers, $P^{(1)}$ is real. From equation (7.98) in Leslie (1983), we have

$$S_1 = -(\kappa/2) \int \{P^{(1)}\}^2 / P^{(F)} \, \mathrm{d}\boldsymbol{X} < 0.$$

The part S_0 of turbulence entropy in (2) corresponds to the Gaussian probability distribution, and the part S_1 represents the correction due to non-Gaussianity. Hence the physical meaning of the above lemma is that the non-Gaussianity will decrease the entropy, which is expected. □

Theorem. The turbulence entropy (2) does not have a maximum value under the constraint (4*a*) or (4*b*).

Proof. Assume that the constraint is (4*a*) and S attains a maximum value at $\eta_i = \bar{\eta}_i$ and $\phi_i = \bar{\phi}_i$. Obviously $\lambda\bar{\eta}_i$ and $\lambda\bar{\phi}_i$ also satisfy the constraint (4*a*) for any positive number λ. Let $S(\lambda)$, $S_0(\lambda)$ and $S_1(\lambda)$ denote the values of S, S_0 and S_1 while $\eta_i = \lambda\bar{\eta}_i$ and $\phi_i = \lambda\bar{\phi}_i$ in (2), then the above assumption about the maximum of S implies that $S(\lambda)$ attains a maximum value at $\lambda = 1$. By (2*b*), $S_0(\lambda)$ increases as λ increases, i.e. $S_0(\lambda) > S_0(1)$ if $\lambda > 1$. From (2*c*) we have $S_1(\lambda) = S_1(1)/\lambda$, and $S_1(1) < 0$ by the lemma, so $S_1(\lambda) > S_1(1)$ if $\lambda > 1$. Hence $S(\lambda) > S(1)$ if $\lambda > 1$, i.e. $S(1)$ is not a maximum value, which contradicts the original assumption. Therefore, S has no maximum under the constraint (4*a*).

Similarly, assume that S attains a maximum value at $\eta_i = \bar{\eta}_i$ and $\phi_i = \bar{\phi}_i$ under the constraint (4*b*). According to the property of $\Pi(k)$ mentioned above, $\lambda^2\bar{\eta}_i$ and $\lambda\bar{\phi}_i$ also satisfy the constraint (4*b*). Let $S(\lambda)$, $S_0(\lambda)$ and $S_1(\lambda)$ denote the values of S, S_0 and S_1 while $\eta_i = \lambda^2\bar{\eta}_i$ and $\phi_i = \lambda\bar{\phi}_i$ in (2), then $S(\lambda)$ attains a maximum value at $\lambda = 1$ by the above assumption. By (2*b*), $S_0(\lambda) > S_0(1)$ if $\lambda > 1$. From (2*c*) we have $S_1(\lambda) = S_1(1)/\lambda^3$, and $S_1(1) < 0$ by the lemma, so $S_1(\lambda) > S_1(1)$ if $\lambda > 1$. Hence $S(\lambda) > S(1)$ if $\lambda > 1$, which contradicts the original assumption. Therefore, in the case of constraint (4*b*) S has no maximum either. As a consequence, we have the following corollary.

Corollary. The solution of the basic equations (2)–(4) of the entropy method does not exist.

The reader might wonder how McComb (1990) can obtain a 'solution' of (2)–(4) which actually have no solution at all and 'successfully' derive the inertial-range spectrum. After a careful examination of his solution, it is found that the variations in η_i and ϕ_i, made by McComb in his derivation, do not satisfy the energy equation. Therefore, McComb's mathematics is not in conformity with the premise of his entropy method that the variation of η_i and ϕ_i are not independent and must be subject to the energy equation.

4. Discussion on variational approach

It is obvious that the value of S is dependent of the choice of variables (Edwards and McComb 1969). The real issue is whether the working expression for S used in the entropy method (equation (2)) has a maximum value while the variables have already been chosen and are subject to the energy equation. The elementary proof given in the above section clearly proves that the working expression for S has no maximum value while the variables have already been chosen and are subject to the energy equation, so the solution of the basic equations of the entropy method does not exist.

It is interesting to discuss McComb's comment on other variational approaches, which was recently made in connection with his efforts to defend the entropy method. McComb

(1990, p 294) said: 'An alternative variational method has been proposed by Quian (1983), but this treats the variation as if $\omega(k)$ and $q(k)$ are independent. In fact, this is the same as if the second term on the right-hand side of our equation (7.88) has been neglected. Hence, it follows that Quian's method is mathematically incorrect.' This author is sorry to say that McComb has mistaken Qian for Quian. McComb's $\omega(k)$ and $q(k)$ correspond to η_i and ϕ_i of (4a), and his equation (7.88) is the same as (3).

The success of a variational approach depends upon the proper choice of the criterion (or objective function) as well as the constraint condition which define admissible values of the variables or parameters. The entropy method chooses the turbulence entropy as the criterion and the energy equation as the constraint condition. This choice is not proper, since the solution for the basic equations of the entropy method does not exist. A totally different point of view is adopted in the variational approach proposed by Qian (1983), where the criterion is the error $l(\eta)$ of the approximate solution of Liouville equation instead of the turbulence entropy, $\eta = \{\eta_1, \eta_2, \ldots\}$ is a set of control parameters to be adjusted to minimize the error $l(\eta)$ while the modal intensity ϕ_i are given. No one will doubt the legitimacy and necessity of minimizing the error $l(\eta)$ of the approximate solution of Liouville equation. Qian (1983) used $l(\eta)$, instead of $l(\phi, \eta)$, to denote the criterion in order to emphasize that ϕ_i are fixed while adjusting η so there is no variation of ϕ_i. Hence Qian's approach is essentially different from the entropy method, and its variation equation $\delta l(\eta)/\delta\eta = 0$ is absolutely not 'the same as if the second term on the right-hand side of McComb's equation (7.88) had been neglected'.

In McComb (1990), there is variation of $q(k)$ as well as $\omega(k)$, and the admissible values of $\omega(k)$ and $q(k)$ (so the variations $\delta\omega(k)$ and $\delta q(k)$) are dependent and satisfy the energy equation. In Qian (1983), η_i are control parameters to minimize the error $l(\eta)$ while the ϕ_i are given, there is no variation of ϕ_i, and the control parameters η_i play a dual role: admissible values of the control parameters can be any real number and not subject to the energy equation, only their optimal values, which minimizes the error $l(\eta)$, represent the 'real physical quantity' satisfying the energy equation. Hence the variation in Qian's control parameters η_i are not subject to the constraint (4a) or (4b). The dual role of control parameters is self-evident in an optimization problem. It is not proper for McComb to assert that Qian's method 'is mathematically incorrect', since the mathematics used by Qian (1983) is in conformity with the premise of his variational approach that the η_i are treated as control parameters to minimize the error $l(\eta)$ while ϕ_i are given. Of course, Qian's mathematics is certainly not in conformity with McComb's idea of η and his entropy method. In a variational formulation of physical problems, it is allowable that the admissible values of the variables might not conform to physical laws. For example, in the case of variational principle of optics or mechanics, the admissible (or virtual) light or particle paths might not conform to physical laws.

It is easy to prove that Qian's criterion $l(\eta)$, which represent the error of the approximate solution of the Liouville equation, does have a minimum value while the modal intensity ϕ_i are given, so the solution of the variation equation $\delta l(\eta)/\delta\eta = 0$ does exist. In Qian (1983), both the variation equation $\delta l(\eta)/\delta\eta = 0$ and the energy equation are used to determine ϕ_i and η_i, so the resultant η_i, representing optimal value of the control parameter, does satisfy the energy equation. Of course, the admissible values of the control parameter might not satisfy the energy equation. The variational approach proposed by Qian (1983, 1985, 1986, 1990) has been successfully applied to derive the velocity energy spectrum and the scalar variance spectrum; moreover, it can be applied to account for intermittence of small scales and calculate the flatness factor of the velocity derivative. All these theoretical predictions are in agreement with experimental data.

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References

Edwards S F and McComb W D 1969 *J. Phys. A: Math. Gen.* **2** 157

Leslie D C 1983 *Developments in the Theory of Turbulence* (paperback with corrections) (Oxford: Clarendon)

McComb W D 1990 *The Physics of Fluid Turbulence* (Oxford: Clarendon)

Qian J 1983 *Phys. Fluids* **26** 2098; 1985 *Phys. Fluids* **28** 1299; 1986 *Phys. Fluids* **29** 3586; 1990 *J. Fluid Mech.* **217** 203